ON ONE PROBLEM OF FILTRATION WITH A LIMIT GRADIENT ALLOWING AN EXACT SOLUTION

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An exact solution of the problem of a field of an infinite series of sources in the presence of filtration with a limit gradient is derived by the method of integral transformations. The stagnation zone boundary is determined, and the results are compared with those of the approximate solution.

1. It was shown in paper [1] that a series of symmetrical filtration problems with limit gradients is reduced by means of hodograph transformation to the determination of the stream function ψ as a solution of the linear equation

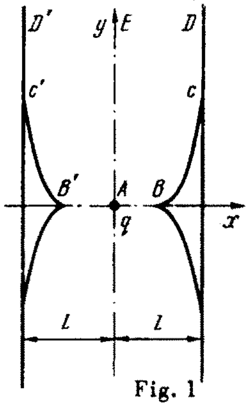

Fig. 1

$$w(w+\lambda)\frac{\partial^2\psi}{\partial w^2}+(w-\lambda)\frac{\partial\psi}{\partial w}+\frac{\partial^2\psi}{\partial\theta^2}=0 \qquad (1.1)$$

where w is the filtration rate modulus, θ its angle to the x-axis, and λ the characteristic initial rate proportional to the initial gradient magnitude at which motion begins.

In these problems Eq. (1.1) has to be solved in the semi-infinite band $0<\theta<\theta_0$, $0<w<\infty$ sectioned along segment $0<w<a$, or half-line $a<w<\infty$, $\theta=\theta_1$ with specified boundary values for ψ along the whole boundary. The simplest limit cases of $a=0$ were considered in paper [1]; the derivation of a solution for $a\neq 0$ is generally not possible. In the following a particular case in which an exact solution is comparatively easily derived is considered. In the physical plane it corresponds to the flow from an infinite row of sources of intensity q equally spaced along a straight line, or to, what is equivalent a flow from a source located between two impermeable boundaries (Fig. 1). The corresponding picture in the $w\theta$-plane is shown on Fig. 2, where $a=\frac{1}{4}q/L$. By virtue of the obvious symmetry of the problem the stream function vanishes along half-line $a<w<\infty$, $\theta=\frac{1}{2}\pi$, hence the problem under consideration is equivalent to the first boundary value problem for the semi-infinite band $0<w<\infty$, $0<\theta<\frac{1}{2}\pi$ with conditions

$$\psi(w,\theta)=\psi(0,\theta)=0$$
$$\psi(w,{}^1/_2\pi)=0\ (0<w<a),\quad \psi(w,{}^1/_2\pi)=Q={}^1/_4 q\ (a<w<\infty) \qquad (1.2)$$

2. It is convenient for solving this problem, as well as for analyzing other problems related to Eq. (1.1), to resort to an integral transformation with respect to variable w. For this it is necessary to establish formulas for the integral expansion with respect to eigenfunctions of Eq.

$$u(u+1)Y''+(u-1)Y'+\alpha Y=0 \qquad (2.1)$$

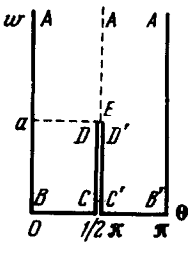

Fig. 2

The general method of expansion derivation is given in Titchmarsh's book [2] where (in Chapter 4) the expansion derivation for the hypergeometric equation

$$u\,(u+1)Y'' + [\gamma + (\zeta - 1)\ u]Y' + \Lambda\, Y = 0 \qquad (2.2)$$

for $\gamma > 2$ is analyzed in detail. In the case of Eq. (2.1) $\gamma = -1$ and the Titchmarsh formulas are not directly applicable. We may however repeat more or less literally the reasoning given in [2], taking into account that in this case out of the two independent solutions of Eq. (2.1) that which conforms to the required behavior in the neighborhood of zero is

$$Y_2 = u^2 F\,(2 + i\sqrt{\alpha},\ 2 - i\sqrt{\alpha},\ 3,\ -u) \qquad (2.3)$$

Here F is the hypergeometric function symbol, $\sqrt{1} = 1$, and the section for the root is taken along the negative semiaxis α. As the result we obtain

$$g\,(u) = {}^1/_4 u^2 \int\limits_0^\infty \alpha\,(1+\alpha)\,\mathrm{cth}\,(\pi\sqrt{\alpha})\,F\,(2 + i\sqrt{\alpha},\ 2 - i\sqrt{\alpha},\ 3,\ -u)\,g^*(\alpha)\,d\alpha \qquad (2.4)$$

The transform $g^*(\alpha)$ is defined by Expression

$$g^*(\alpha) = \int\limits_0^\infty (1+u)\,F\,(\alpha,\ u)\,g\,(u)\,du, \quad F\,(\alpha,\ u) = F\,(2 + i\sqrt{\alpha},\ 2 - i\sqrt{\alpha},\ 3,\ -u) \qquad (2.5)$$

3. We assume $u = w/\lambda$ in Eq. (1.1) which then becomes

$$u\,(u+1)\,\frac{\partial^2\psi}{\partial u^2} + (u-1)\,\frac{\partial\psi}{\partial u} + \frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad (3.1)$$

and apply to it transformation (2.5), i.e. multiply it by $(1+u)\ F\ (\alpha,\ u)$, and then integrate with respect to u from 0 to ∞. Integrating by parts and assuming the contribution of terms outside of the integral to be nil, we obtain

$$\frac{d^2\psi^*(\alpha,\ \theta)}{d\theta^2} - \alpha\psi^*(\alpha,\ \theta) = -\,2\psi\,(0,\ \theta) \qquad (3.2)$$

In this case the right-hand side of Eq. (3.2) vanishes by virtue of the condition $u = 0$, and the solution which satisfies condition for $\theta = 0$ is of the form

$$\psi^*\,(\alpha,\ \theta) = A\,(\alpha)\,\mathrm{sh}\,\sqrt{\alpha}\,\theta \qquad (3.3)$$

Applying transformation (2.5) to the boundary condition (1.2) with $\theta = {}^1\!/_2\,\pi$, and using the conversion formula (2.4), we obtain a solution of the form

$$\psi\,(u,\ \theta) = \frac{Qu^2}{4}\int\limits_0^\infty \frac{\alpha\,(1+\alpha)\,\mathrm{cth}\,\pi\sqrt{\alpha}\,\mathrm{sh}\,\sqrt{\alpha}\,\theta}{\mathrm{sh}\,{}^1/_2\,\pi\sqrt{\alpha}}\,F\,(\alpha,\ u)\int\limits_{a_0}^\infty (1+v)\,F\,(\alpha,\ v)\,dv\,d\alpha,\quad a_0 = a/\lambda \qquad (3.4)$$

We note that by virtue of the following identity (see, e.g. [3 and 4])

$$F\,(\alpha,\ u) = u^{-2-i\sqrt{\alpha}}\,\frac{\Gamma(3)\,\Gamma(-2i\sqrt{\alpha})}{\Gamma(1 - i\sqrt{\alpha})\,\Gamma(2 - i\sqrt{\alpha})}\,F_+(-1/u) +$$
$$+\, u^{-2+i\sqrt{\alpha}}\,\frac{\Gamma(3)\,\Gamma(2i\sqrt{\alpha})}{\Gamma(1 + i\sqrt{\alpha})\,\Gamma(2 + i\sqrt{\alpha})}\,F_-(-1/u)$$

$$F_{\pm}(-1/u) = F(2 \pm i\sqrt{\alpha},\ \pm i\sqrt{\alpha},\ 1 \pm 2i\sqrt{\alpha},\ -1/u) \tag{3.5}$$

$$F(\alpha, u) = u^{-2}(C_1 u^{-i\sqrt{\alpha}} + C_2 u^{i\sqrt{\alpha}})(1 + o(1)) \quad \text{for} \quad u \to \infty$$

Hence the inner integral in (3.4) is convergent when $\alpha \neq 0$, while for $\alpha = 0$ it has a root singularity.

4. Using solution (3.4) we shall determine the stagnation zone boundary form (image of segment $w = 0$, $0 < \theta < \frac{1}{2}\pi$). For this it is necessary according to [1] to calculate magnitude

$$\chi(\theta) = \lambda \lim_{w \to 0} \frac{1}{w} \frac{\partial \psi}{\partial w} = \frac{1}{\lambda} \lim_{u \to 0} \frac{1}{u} \frac{\partial \psi}{\partial u} \tag{4.1}$$

From (3.4) we have

$$\chi(\theta) = \frac{Q}{2\lambda} \int_0^\infty \frac{\alpha(1+\alpha)\operatorname{cth}\pi\sqrt{\alpha}\operatorname{sh}(\theta\sqrt{\alpha})}{\operatorname{sh}{}^1/{}_2\pi\sqrt{\alpha}} \int_{a_1}^\infty (1+v) F(\alpha, v)\, dv\, d\alpha \tag{4.2}$$

In order to transform the integral with respect to α we shall use relationship (3.5). The integrand of (4.2) will be split into the sum of two terms, one of which is analytical and decreases exponentially in the upper half-plane with increasing $|\alpha|$, while the other does so in the lower half-plane. (This can be ascertained with the aid of Watson's formulas in Section 2 of reference book [4]).

Integral (4.2) is thus split into two; the integration path of the first one may be distorted along the negative real semiaxis α into the cross section upper edge which is penetrated from right to left, bypassing in the upper half-plane points $\alpha = -n^2$ along infinitely small semicircles (outline Γ_+). The second integral may be similarily computed along the cross section lower edge bypassing points $\alpha = -n^2$ from below along small semicircles (outline Γ_-), so that

$$\chi(\theta) = -\frac{Q}{2\lambda} \int_{\Gamma_+} \frac{\alpha(1+\alpha)\operatorname{cth}\pi\sqrt{\alpha}\operatorname{sh}\theta\sqrt{\alpha}}{\operatorname{sh}{}^1/{}_2\pi\sqrt{\alpha}} \frac{\Gamma(3)\Gamma(2i\sqrt{\alpha})}{\Gamma(1+i\sqrt{\alpha})\Gamma(2+i\sqrt{\alpha})} \times$$
$$\times \int_{a_1}^\infty (1+v) F_-(-1/v)\, dv d\alpha + \frac{Q}{2\lambda} \int_{\Gamma_-} \frac{\alpha(1+\alpha)\operatorname{cth}\pi\sqrt{\alpha}\operatorname{sh}(\theta\sqrt{\alpha})}{\operatorname{sh}{}^1/{}_2\pi\sqrt{\alpha}} \times$$
$$\times \frac{\Gamma(3)\Gamma(-2i\sqrt{\alpha})}{\Gamma(1-i\sqrt{\alpha})\Gamma(2-i\sqrt{\alpha})} \int_{a_1}^\infty (1+v) F_+(-1/v)\, dv\, d\alpha$$

We now note that the integrand of the first integral considered as a function of α is analytical in the upper half-plane, while that of the second integral is analytical in the lower one, and that along segments $(-n^2, -(n-1)^2)$ of the real axis the two coincide by virtue of condition $\sqrt{\alpha} = i\sqrt{|\alpha|}$ above, and $\sqrt{\alpha} = -i\sqrt{|\alpha|}$ below the latter. Hence, by virtue of the principle of analytic extension, these form an analytical function $\Xi(\alpha)$, which may possibly have points $\alpha = -n^2$ as its poles. Thus we have

$$\chi(\theta) = \frac{Q}{2\lambda} \int_\Gamma \Xi(\alpha)\, d\alpha \qquad (\Gamma = \Gamma_+ + \Gamma_-) \tag{4.3}$$

By virtue of the above the integrals cancel each other out along segments $(-n^2, -(n-1)^2)$ and Expression (4.3) is reduced to the sum of integrals along the infinitely small circles, bypassing points $\alpha = -n^2$. (We note that the singular points (poles) of function $F_{\pm}$ are cancelled by zeros of the hyperbolic cotangent, hence their contribution is nil). The contribution of those of the $\alpha = -n^2$ points which correspond to odd $n = 2m + 1$ is also nil. At such points we have, in fact, for $m = 0$ a pole of the first order, and poles at $\Gamma(-2i\sqrt{\alpha})$, and $\Gamma(1 - i\sqrt{\alpha})$; here $(1 + \alpha)$ also vanishes, hence a singularity is absent.

An extraneous pole appears in the denominator when $m \neq 0$, so that again there is no singularity. We shall now consider points $n = 2m$. When $m = 0$ the inner integral is of the order of

$$\int_{a_1}^{\infty} u^{-1+i\sqrt{\alpha}}\, du = O(1/\sqrt{|\alpha|})$$

The expression in front of it is of the order of unity, hence the singularity at point $\sqrt{\alpha}$ is integrable, and its contribution nil. If $m \neq 0$, then $h\ {}^1\!/\!_2\,\pi\sqrt{\alpha}$, vanished, the pole has a cotangent and all of the gamma-functions, therefore the complete integrand has poles of the first order at points $\alpha = -(2m)^2$. After computation we obtain

$$\chi(\theta) = \frac{16Q}{\pi\lambda} \sum_{m=1}^{\infty} (-1)^m m^2 \frac{(2m-1)!\,(2m+1)!}{4m!} \times$$

$$\times \int_{a_1}^{\infty} (1+v)\, v^{-2-2m} F(2+2m,\, 2m,\, 1+4m,\, -1/v)\, dv \sin 2m\theta \qquad (4.4)$$

Using, as in [1], the Euler representation for the hypergeometric function, we can express (4.4) in the form

$$\chi(\theta) = \frac{16Q}{\pi\lambda} \int_{a_1}^{\infty} \frac{1+u}{u^2}\, du \int_0^1 \operatorname{Im} \frac{t^3}{(t+u)^2} \frac{e^{2i\theta}(1 - 8\nu^2 e^{2i\theta} + 3\nu^4 e^{4i\theta})}{(1+\nu^2 e^{2i\theta})^4}\, dt \left(\nu = \frac{t(1-t)}{u+t}\right) \qquad (4.5)$$

For the coordinates of the stagnation zone boundary we have

$$x(\varphi) + iy(\varphi) = \int_0^{\varphi} e^{i\theta} \chi(\theta)\, d\theta \qquad (4.6)$$

Substituting (4.5) into (4.6) and integrating with respect to θ, we obtain

$$x(\varphi) + iy(\varphi) = -\frac{8Q}{\pi\lambda} \int_{a_0}^{\infty} \frac{1+u}{u^2}\, du \int_0^1 \frac{t^3 dt}{(t+u)^2} \left[\frac{1}{4\nu^3} \operatorname{arctg} \frac{\nu\cos\varphi}{1+\nu\sin\varphi} + \right.$$

$$+ \frac{1}{4\nu^3} \operatorname{arctg} \frac{\nu\cos\varphi}{1-\nu\sin\varphi} - \frac{1}{2\nu^3} \operatorname{arctg} \nu - \frac{A(\nu,\varphi)\cos\varphi - B(\nu,\varphi)\sin\varphi}{2\nu^2(1+2\nu^2\cos 2\varphi + \nu^4)^3} +$$

$$\left. + \frac{1+7\nu^4}{2\nu^2(1+\nu^2)^3} + \frac{i}{8\nu^3} \ln \frac{1+2\nu\sin\varphi+\nu^2}{1-2\nu\sin\varphi+\nu^2} - i\, \frac{A(\nu,\varphi)\sin\varphi + B(\nu,\varphi)\cos\varphi}{2\nu^2(1+2\nu^2\cos 2\varphi+\nu^4)^3} \right] \qquad (4.7)$$

Here

$$A(\nu,\varphi) = 1 + 3\nu^2\cos 2\varphi + 10\nu^4\cos 4\varphi + 21\nu^6\cos 2\varphi + \nu^6\cos 6\varphi + 21\nu^8 + 7\nu^{10}\cos 2\varphi$$

$$B(\nu,\varphi) = \nu^2\sin 2\varphi - 3\nu^6\sin 2\varphi - \nu^6\sin 6\varphi - 4\nu^8\sin 4\varphi - 3\nu^{10}\sin 2\varphi$$

Expression (4.7) defines the boundary conditions to within the constants which are easily defined by taking into account conditions $x({}^1\!/\!_2\,\pi) = L$, $y(0) = 0$.

Outline of the stagnation zone boundary computed from Formula (4.7) is shown on Fig. 3, where the figures adjacent to curves denote values of parameter a_0.

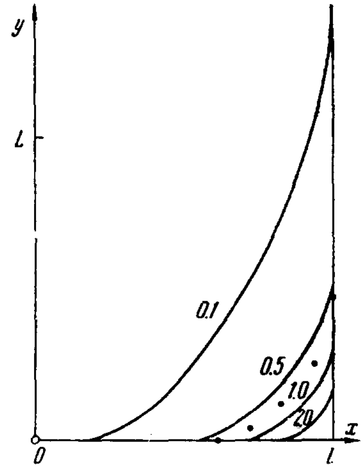

Fig. 3

5. It is important in practice to know in addition to the disposition of stagnation zones the effect of their onset on the pressure drop between the slot and the layer distant parts.

We select on the y-axis (Fig. 1) points $y_1 \ll L$ and $y_2 \gg L$. In accordance with the law of filtration we have

$$\frac{\partial H}{\partial y} = -(w + \lambda) \tag{5.1}$$

hence

$$H(y_1) - H(y_2) = \lambda (y_2 - y_1) + \int\limits_{y_1}^{y_2} w\, dy \tag{5.2}$$

The solution of this problem for $\lambda = 0$ is well known (see, e.g., [5]). In particular, the filtration rate distribution along the y-axis is defined by Expression

$$w^\circ(0, y) = \frac{Q}{L} \operatorname{cth} \frac{\pi y}{2L} \tag{5.3}$$

Using (5.3) and disregarding small magnitude we obtain from (5.2)

$$H(y_1) - H(y_2) = \lambda (y_2 - y_1) + \Delta H_D + \int\limits_0^\infty \left(w - \frac{Q}{L} \operatorname{cth} \frac{\pi y}{2L} \right) dy \tag{5.4}$$

$$H_D = \frac{Q}{L} \int\limits_{y_1}^{y_2} \operatorname{cth} \frac{\pi y}{2L}\, dy \tag{5.5}$$

Here H_D is the pressure drop corresponding to the linear law of filtration, $\lambda(y_2 - y_1)$ represent an additional pressure loss allied to the nonlinearity of the filtration law. Additions of this kind would be obtained in the case of flows from a straight line manifold. The integral term of (5.4) takes into account variation of the pressure loss resulting from the change of the flow pattern under the influence of the flow nonlinearity. In its computation the relationship between w and y is defined by Expression [1]

$$y = -\int\limits_w^\infty \frac{\partial H}{\partial w}\bigg|_{\theta = {}^1\!/_2\pi} \frac{dw}{w + \lambda} = -\int\limits_w^\infty \frac{\partial \psi}{\partial \theta}\bigg|_{\theta = {}^1\!/_2\pi} \frac{dw}{w^2} \tag{5.6}$$

$$Q/L \leqslant w < \infty$$

The expression of derivative $\partial\psi/\partial\theta$ is obtained by differentiation (3.4), and in the general case is fairly complicated. It can be simplified for large and small values of $a_0 = Q/\lambda L$.

6. We shall now consider the limit case when $a = 0$. As was shown in [1] the solution yields in this limit case the lower estimate for the stagnation zone dimensions, it is therefore of interest to compare it with the solution derived above for small $a \neq 0$. The required solution may be taken directly from paper [1].

We have

$$\psi = (2/\pi)Q\theta + \Psi^\circ \tag{6.1}$$

$$\Psi^\circ = -\frac{2Q}{\theta_1(1+u)}\,\mathrm{Im}\int_0^1 \frac{1+u-t}{t}\,\frac{(1+\pi/\theta_1)\,\nu^{\pi/\theta_1}e^{i\theta\pi/\theta_1} + \nu^{2\pi/\theta_1}e^{2i\pi\theta/\theta_1}}{(1+\nu^{\pi/\theta_1}e^{i\pi\theta/\theta_1})^2}\,dt \tag{6.2}$$

(In the corresponding formula (4.23) of paper [1] the exponent in denominator had been inadvertently omitted, and the numerator incorrectly written). Assuming here $\theta_1 = \frac{1}{2}\pi$ we calculate

$$x^\circ = \frac{1}{\lambda}\int_0^\infty \frac{\partial\psi(u,\theta)}{\partial\theta}\Big|_{\theta=0}\frac{du}{u^2} = \frac{8Q}{\pi\lambda}\int_0^\infty\left[\frac{1}{4} - \frac{1}{1+u}\int_0^1 \frac{(1-t)\,\nu\,(3-\nu^2)}{(1+\nu^2)^3}\,dt\right]\frac{du}{u^2} \tag{6.3}$$

The last integral may be integrated by parts, followed by integration with respect to ν and then the order of integration changed which allows the finding of

$$\frac{\pi\lambda x^\circ}{16Q} = \int_0^1\left\{\frac{\nu\,[6-3\nu-16\nu^2-2\nu^3+2\nu^4+\nu^5]}{(1+\nu^2)^4}\left[\frac{1-\nu]}{(1+\nu)^2} - \frac{2\nu\ln\nu}{(1+\nu)^3}\right]\right\}d\nu = 0.45 \tag{6.4}$$

In order to determine the form of the stagnation zone we compute

$$\lim_{u\to 0}\frac{1}{u}\frac{\partial\psi}{\partial u} = \frac{24Q}{\pi}\sin 2\theta\left(\frac{\pi}{16\cos^5\theta} + \frac{1}{6\cos^4\theta} + \frac{1}{24\cos^2\theta} + \right.$$
$$\left. + \frac{\theta}{8\sin\theta\cos^5\theta} - \frac{\theta}{16\sin\theta\cos^3\theta}\right) \equiv \lambda\chi_0(\theta) \tag{6.5}$$

From this we derive the expression for the stagnation zone boundary

$$x(\theta) + iy(\theta) = x^\circ + \int_0^\theta \chi_0(\theta)\,e^{i\theta}\,d\theta$$

or

$$\begin{aligned} x(\theta) &= x^\circ + \frac{24Q}{\pi\lambda}\left[\frac{\pi}{16}\,\mathrm{tg}^2\theta + \frac{5}{24}\,\frac{1-\cos\theta}{\cos\theta} + \frac{1-\cos\theta}{12} + \frac{1}{8}\,\frac{\theta\sin\theta}{\cos^2\theta}\right] \\ y(\theta) &= \frac{24Q}{\pi\lambda}\left[\frac{\pi}{24}\,\mathrm{tg}^3\theta + \frac{1}{8}\,\frac{\sin\theta}{\cos^2\theta} - \frac{1}{12}\sin\theta + \frac{\theta}{12\cos^3\theta} - \frac{\theta}{8\cos\theta}\right] \end{aligned} \tag{6.6}$$

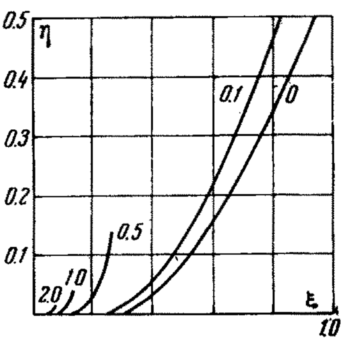

Fig. 4

The stagnation zone limit pattern defined by Expressions (6.5) and (6.6) is shown on Fig. 4, on which the curves of Fig. 3 have also been drawn with magnitude Q/λ as the scale ($\xi = \pi\lambda x/(24\,Q)$, $\eta = \pi\lambda y/(24\,Q)$) for the sake of comparison. As expected, the limit solution ($a_0 = 0$) yields the lower estimate of the stagnation zone dimensions, while for $a_0 = 0.1$ this estimate is already close to the solution.

7. An approximate method was proposed in paper [6] for solving the problem of filtration with an initial gradient which amounts to the assumption that motion in the flow area at a velocity exceeding the stipulated

value of λ conforms to the Darcy law, while the stagnation zone boundaries are found as streamlines along which the filtration rate is equal to λ. Thus stated the problem is reduced to the solution of the Laplace equation in the area with an unknown boundary, and is readily solved by methods known in the theory of jets.

In the case here considered it is necessary to find a solution of the Laplace equation for the stream function ψ in the area $EABCD$ (Fig. 1), having satisfied the complementary condition that $w = \lambda$ along BC. The required expression binding z by the potential W is of the form

$$\frac{dz}{dW} = \frac{i(1+a_0^2)}{2a_0\lambda}\left[\sqrt{1-\exp(-\pi W/Q)} - \sqrt{\alpha^2-\exp(-\pi W/Q)}\right]$$
$$\alpha = (a_0^2-1)/(a_0^2+1) \qquad (W = H + i\psi,\ z = x + iy) \tag{7.1}$$

Integrating and assuming that $\psi = 0$, we obtain the looked for expression of the stagnation zone boundary. If angle θ between the boundary and the x-axis is taken as a parameter then

$$\frac{z}{L} = \frac{a_0^2-1}{\pi}\operatorname{arctg}\frac{2a_0\cos\theta}{a_0^2-1} - \frac{2a_0}{\pi}\cos\theta +$$
$$+ i\left[\frac{1+a_0^2}{2\pi}\ln\frac{1+2a_0\sin\theta+a_0^2}{1-2a_0\sin\theta+a_0^2} - \frac{2}{\pi}a_0\sin\theta\right] \tag{7.2}$$

It will be readily seen that the obtained solution holds for sufficiently great a_0 only. In fact, when $a_0 < 1$ the stream velocity at infinity is lower than λ, which is contrary to the model here considered; for a_0 close to unity the stagnation zone tip receeds arbitrarily for along the y-axis. Boundary coordinates calculated by means of Formula (7.2) for $a_0 = 2$ are shown by dots on Fig. 3. It is evident that the result differs considerably from that of the exact solution.

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